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APPROXIMATION OF NONSTATIONARY PROCESSES ON AN INFINITE TIME INTERVAL FOR EXPONENTIAL STABILITY OF SLOW MOTIONS

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Systems in standard form and quasilinear systems with many fast variables are examined. It is shown that when the transient slow motions are uniformly exponentially stable, the solutions approximate the exact ones on an infinite time interval under an asymptotic separation of the motions. A relation is found between the size of the stability domain, the order of the exponent in the estimate of the resolving matrix of the equations in variations, and the number of operations ensuring the approximation.

1. Asymptotic approximation of nonquasistationary solutions of systems in standard form on an infinite time interval. In an n-dimensional Euclidean space $x_1, \ldots, x_n$ we consider a system in standard form

$$x^{\cdot} = \varepsilon X(x, t, \varepsilon) \tag{1.1}$$

where x is a column vector and the vector-valued function X, for $t \geqslant t_0$, $|\varepsilon| \leqslant \varepsilon_0$ and x from some domain G, is continuous in t and is uniformly bounded together with $m+1$ derivatives with respect to x and m derivatives with respect to ε. Let us consider an improved m-th approximation to the solution of system (1.1), constructed by the averaging method

$$x^{(m)} = \xi_m + \varepsilon u_1(t, \xi_m) + \ldots + \varepsilon^m u_m(t, \xi_m) \tag{1.2}$$

where ξ_m satisfies the equation

$$d\xi_m / dt = \varepsilon \Xi_0(\xi_m) + \ldots + \varepsilon^m \Xi_{m-1}(\xi_m) = \varepsilon \Xi^{(m-1)}(\xi_m, \varepsilon) \tag{1.3}$$

We assume that $u_j(t_0, \xi_m) = 0$, $j = 1, \ldots, m$. Then ξ_m should be determined under the initial condition $\xi_m(t_0) = \xi_{m0} = x^{(m)}(t_0) = x(t_0)$. We assume that all the means of form

$$\Xi_j = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} U_j(t, \xi_m)\, dt \tag{1.4}$$

encountered during computations by the averaging method, exist and are uniformly bounded in domain G together with the first derivatives with respect to ξ_m, and that the functions u_j and $\partial u_j / \partial \xi_m$ are uniformly bounded for $t \geqslant t_0$, $\xi_m \in G$.

It was shown in [1] that the bound $|x - x^{(m)}| \leqslant C_m \cdot |\varepsilon|^m$ is valid for solutions of form (1.2) on an interval of order $T / |\varepsilon|$, independently of the properties of the evolutionary components $\xi_m(t)$. The same bound is possible on intervals of larger order [2] for quasistationary solutions $\xi_m = \text{const}$ and on an infinite interval for stable quasistationary solutions (Bogoliubov's theorem). Later on we indicate some other cases of existence of a uniform approximation of nonstationary motions on an infinite interval both for systems in standard form as well as for systems of a more general form.

By $\xi_m(t, t_0, a)$ we denote the solution of Eq. (1.3) with initial condition $\xi_m(t_0) = \xi_{m0} = a$, and by $U_m(t, s, t_0, a)$, the solution of the matrix equation

$$\frac{dU_m}{dt} = \varepsilon\left[\frac{d}{d\xi_m}\,\Xi^{(m-1)}(\xi_m, \varepsilon)\big|_{\xi_m=\xi_m(t,\,t_0,\,a)}\right]U_m \tag{1.5}$$

with initial condition $U_m(s, s, t_0, a) = E$, where E is the unit matrix; $U_m(t, s, t_0, a)$ is the resolving matrix of the equation in variations of systems (1.3) on the solution $\xi_m(t, t_0, a)$. Later on we examine the case when the solution $\xi_m(t, t_0, a)$ is uniformly exponentially stable with respect to the linear approximation, i.e., when

$$\|U_m(t, s, t_0, a)\| \leqslant N_m e^{-\nu_m(t-s)} \tag{1.6}$$

for all $t \geqslant s \geqslant t_0$.

T h e o r e m 1. Suppose that the solution $\xi_m(t, t_0, a)$, together with its ρ-neighborhood, where ρ is independent of t and ε, remains in G for all $t \geqslant t_0$ and for sufficiently small $|\varepsilon|$. Let relation (1.6) be valid and let the exponential stability first appear in terms of the k-th order in ε, i.e., $\nu_m = |\varepsilon|^k \nu_{mk}$, ν_{mk} and N_m being independent of ε. Let the solution $\xi_m(t, t_0, a)$ belong to a family of uniformly exponentially stable solutions of the following form: for all $t_1 \geqslant t_0$ there exists a ball of radius δ not depending on t_1, with center at point $\xi_m(t_1, t_0, a)$, such that each solution $\xi_m(t, t_1, \xi_{m1})$, having hit the point ξ_{m1} of this ball at $t = t_1$, remains in G together with its $\rho(\xi_{m1})$-neighborhood, and to this solution there corresponds a resolving matrix $U_m(t, s, t_1, \xi_{m1})$ satisfying condition (1.6)with the same constants N_m and ν_m when $t \geqslant s \geqslant t_1$. Let the ball's radius be of order r, i.e., $\delta = |\varepsilon|^r \delta_r$, where δ_r is independent of ε. Let $m > k + r - 1$. Then for sufficiently small $|\varepsilon|$ the solution $x(t)$ of the original system (1.1) with initial condition $x(t_0) = a$ remains in G for all $t \geqslant t_0$ and is approximated by the m-th approximation (1.2) on the whole interval $t_0 \leqslant t < \infty$ with accuracy $|\varepsilon|^{m-k+1}$, i.e., for all $t \geqslant t_0$

$$|x(t) - x^{(m)}(t)| < C_m |\varepsilon|^{m-k+1} \tag{1.7}$$

where C_m is independent of t and ε.

P r o o f. In (1.1) we introduce a new variable ξ by the relation

$$x = \xi + \varepsilon u_1(t, \xi) + \ldots + \varepsilon^m u_m(t, \xi) \tag{1.8}$$

For the solution with initial condition $x(t_0) = a$ we have $\xi(t_0) = a$. Since $a \in G - \rho$, an interval $t_0 \leqslant t \leqslant t_0 + T$ exists in which $\xi \in G$. Since the derivatives $\partial u_i / \partial\xi$ are uniformly bounded when $\xi \in G$, a number $\varepsilon_1 \leqslant \varepsilon_0$ exists such that when $|\varepsilon| \leqslant \varepsilon_1$ the matrix $E + \varepsilon\,\partial u_1/\partial\xi + \ldots + \varepsilon^m \partial u_m/\partial\xi$ has an inverse. Under these conditions ξ satisfies the equation

$$d\xi / dt = \varepsilon\Xi^{(m-1)}(\xi, \varepsilon) + \varepsilon^{m+1} R_m(\xi, t, \varepsilon) \tag{1.9}$$

For $\xi \in G$, $|\varepsilon| \leqslant \varepsilon_1$ and $t \geqslant t_0$ the function R_m is uniformly bounded, i.e., the bound $|R_m(\xi, t, \varepsilon)| \leqslant r_m$ is valid, where r_m does not depend on ξ, t, ε.

Equation (1. 9) is equivalent [3] to the integral equation

$$\xi(t, t_0, a) = \xi_m(t, t_0, a) + \varepsilon^{m+1} \int_{t_0}^{t} U_m(t, s, s, \xi(s, t_0, a)) R_m(\xi, s, \varepsilon) ds \quad (1.10)$$

From the condition $\xi - \xi_m = 0$ when $t = t_0$ and the assumption $\xi_m (t, t_0, a) \in G - \rho$ it follows that a time interval $t_0 \leqslant t \leqslant t_0 + T_1$ exists when $| \xi - \xi_m | < \delta$. The relation $\delta < \rho$ is valid by the definition of the quantities ρ and δ ; consequently, $\xi (t) \in G$ when $t_0 \leqslant t \leqslant t_0 + T_1$. In this interval, according to the theorem's statement, the bound (1. 6) is valid for the matrix U_m in the integrand of (1. 10). Therefore,

$$| \xi - \xi_m | \leqslant | \varepsilon |^{m+1} \int_{t_0}^{t} N_m e^{-\nu_m (t-s)} r_m ds < D_m | \varepsilon |^{m-k+1} \quad (1.11)$$

$$D_m = r_m N_m / \nu_{mk}$$

Relation (1. 11) is valid for those t when $| \xi - \xi_m | \leqslant \delta = \delta_r | \varepsilon |^r$. If $m > k + r - 1$, then the inequality $D_m | \varepsilon |^{m-k+1} < \delta$ is fulfilled for sufficiently small $| \varepsilon |$ independently of the values of constants N_m, r_m, ν_{mk}. The equalities $| \xi - \xi_m | = \delta$ and $| \xi - \xi_m | = \rho$ are impossible, and relation (1. 11) is valid for all $t \geqslant t_0$.

Consider relation (1. 8). From the uniform boundedness of functions $u_j (t, \xi)$ for $t_0 \leqslant t < \infty$ and $\xi \in G$ follows

$$| x - \xi | \leqslant | \varepsilon | c_1 + \ldots + | \varepsilon |^m c_m \quad (1.12)$$

where $c_1, \ldots, c_m$ are constants not depending on t, ξ, ε, such that $| u_j (t, \xi) | \leqslant c_j$. But, according to (1. 11), the curve $\xi (t, t_0, a)$ remains in a small neighborhood of curve $\xi_m (t, t_0, a)$ and, consequently, remains in G together with its ρ_1 - neighborhood, where ρ_1 is independent of ε. Therefore, for sufficiently small $| \varepsilon |$ the curve $x (t)$, remaining according to (1. 12) in a small neighborhood of curve $\xi (t)$, remains in G for $t \geqslant t_0$. Analogously with the aid of (1. 2) we can show that $x^{(m)} (t) \in G$ for $t \geqslant t_0$.

Let us estimate $| x - x^{(m)} |$. We have

$$\begin{aligned} | x - x^{(m)} | = {} & | (\xi - \xi_m) + \varepsilon [u_1 (t, \xi) - u_1 (t, \xi_m)] + \\ & \ldots + \varepsilon^m [u_m (t, \xi) - u_m (t, \xi_m)] | \leqslant \\ & | \xi - \xi_m | (1 + | \varepsilon | d_1 + \ldots + | \varepsilon |^m d_m) = \\ & | \xi - \xi_m | (1 + | \varepsilon | d^{(m)}) \end{aligned} \quad (1.13)$$

Here the constants $d_1, \ldots, d_m$, not depending on t and ε, are such that $\| \partial u_j / \partial \xi \| \leqslant d_j$, $t \geqslant t_0$, $\xi \in G$. The existence of such constants follows form the uniform boundedness of derivatives $\partial u_j / \partial \xi$. From (1. 11) and (1. 13) we get that for sufficiently small $| \varepsilon |$

$$| x - x^{(m)} | < D_m (1 + | \varepsilon | d^{(m)}) | \varepsilon |^{m-k+1} < C_m | \varepsilon |^{m-k+1} \quad (1.14)$$

where C_m is a constant not depending on t and ε.

N o t e 1. We can obtain an approximation of the same order by retaining in (1. 2) only the terms containing ε to powers no higher than $m - k$. However, the remaining vibration terms up to degree $m-1$ are necessary for the construction of the functions $\Xi_{m-k+1}, \ldots, \Xi_{m-1}$.

Note 2. The relation

$$U_m(t, t_0, t_0, \xi_{m0}) = \frac{\partial}{\partial \xi_{m0}} \xi_m(t, t_0, \xi_{m0}) \tag{1.15}$$

is well known. Hence it follows that when $|\xi_{m0} - a| \leqslant \delta$

$$|\xi_m(t, t_0, \xi_{m0}) - \xi_m(t, t_0, a)| \leqslant \max_{|\xi_{m0}-a|\leqslant\delta} \|U_m(t, t_0, t_0, \xi_{m0})\| \, |\xi_{m0} - a| \leqslant N_m e^{-\nu_m(t-t_0)} |\xi_{m0} - a| \tag{1.16}$$

i.e., the functions $\xi_m(t, t_0, t_{m0})$ come together as $t \to \infty$. The functions $\xi_m(t, t_1, \xi_{m1})$, falling at $t = t_1$ into a ball of radius δ with center at point $\xi_m(t_1, t_0, a)$, also come together with them. Thus, at each instant $t_1 \geqslant t_0$ the quantity δ is an estimate of the domain of exponential attraction of the solution $\xi_m(t, t_1, \xi_m(t_1, t_0, a))$ with given values N_m and ν_{m_k}. A bound of the form $|\xi(t, t_0, \xi_{m0}) - \xi_m(t, t_0, a)| < D_m |\varepsilon|^{m-k+1}$ is valid, obviously, for the solutions with initial conditions ξ_{m0}, $|\xi_{m0} - a| \leqslant \delta_1 = |\varepsilon|^r \delta_{1r}$, where $\delta_{1r} < \delta_r$ and δ_{1r} is independent of ε. Therefore, the functions $\xi(t, t_0, \xi_{m0})$, remaining in a neighborhood of the functions coming together, will differ from each other for large t by a quantity of order $|\varepsilon|^{m-k+1}$; the same is true of functions $x(t, t_0, \xi_{m0})$. This property can be looked upon as a practical analog of stability.

Generally speaking, the quantities δ and ν_m are independent characteristics of the stability of motion $\xi_m(t, t_0, a)$, which enables up to adopt independent bounds for them. But inequality (1.11) can be obtained without making any assumptions on the magnitude of δ.

Theorem 2. Let function $\xi_m(t, t_0, a)$ and matrix $U_m(t, s, t_0, a)$ satisfy the hypotheses of Theorem 1. Let function $\Xi^{(m-1)}$ have uniformly bounded second derivatives in G. Let $m > 2k - 2$. Then bound (1.17) is valid.

To prove this we make only formal estimates, not proving that the functions being examined lie in G. We estimate the difference $Z_m = \xi - \xi_m$, setting up a nonlinear equation analogous to the Riccati equation (see [3], part 2)

$$dZ_m / dt = \varepsilon [\Xi^{(m-1)}(\xi_m + Z_m) - \Xi^{(m-1)}(\xi_m)] + \varepsilon^{m+1} R_m(Z_m, \xi_m, t, \varepsilon) \tag{1.17}$$

with initial condition $Z_m(t_0) = 0$. We write (1.17) as

$$\frac{dZ_m}{dt} = \varepsilon \left(\frac{\partial \Xi^{(m-1)}}{\partial \xi_m} \right) Z_m + \varepsilon F_m(\xi_m, Z_m) + \varepsilon^{m+1} R_m \tag{1.18}$$

Here the derivative is taken with $\xi_m = \xi_m(t, t_0, a)$. From the uniform boundedness in G of the second derivatives f function $\Xi^{(m-1)}$ follows a bound for the nonlinear term: $|F_m| \leqslant M_m |Z_m|^2$.

Equation (1.18) together with the initial condition is equivalent to the integral equation

$$Z_m = \varepsilon \int_{t_0}^{t} U_m(t, s, t_0, a) [F_m + \varepsilon^m R_m] \, ds \tag{1.19}$$

Using the bound given above, we obtain the integral inequality

$$| Z_m (t) | \leqslant | \varepsilon | I (Z_m) \tag{1.20}$$

$$I (Z_m) = \int_{t_0}^{t} N_m e^{-\nu_m (t-s)} [M_m | Z_m (s) |^2 + | \varepsilon |^m r_m] \, ds$$

Let z_m be a solution of the integral equation

$$z_m (t) = | \varepsilon | I (z_m) \tag{1.21}$$

Then $| Z_m | \leqslant z_m$ (for example, see inequality (1. 25) in Chapter 1 of [4]; in order to make use of this inequality we need to multiply both sides of (1. 20) and (1. 21) by $\exp \nu_m t$). Function z_m satisfies the differential equation

$$dz_m / dt = -\nu_m z_m + | \varepsilon | N_m M_m z_m^2 + | \varepsilon |^{m+1} N_m r_m \tag{1.22}$$

with initial condition $z_m (t_0) = 0$. Function z_m remains bounded for all t if the inequality

$$\nu_m^2 - 4 N_m^2 M_m r_m | \varepsilon |^{m+2} > 0 \tag{1.23}$$

is fulfilled. Since $\nu_m = \nu_{mk} | \varepsilon |^k$, when $m > 2k - 2$ inequality (1. 23) is fulfilled for sufficiently small $| \varepsilon |$ independently of the values of ν_{mk}, N_m, etc. Solving Eq. (1. 22) under condition (1. 23), we obtain

$$z_m (t) < \frac{\nu_{mk} | \varepsilon |^{k-1}}{2 M_m N_m} - \left(\frac{\nu_{mk}^2 | \varepsilon |^{2k-2}}{4 N_m^2 M_m^2} - \frac{r_m | \varepsilon |^m}{M_m} \right)^{1/2} \tag{1.24}$$

Consequently, a constant D_m, independent of t and ε, exists such that bound (1.11) is valid. The proof that $\xi, x \in G$ and the proof of bound (1. 7) are carries out as in Theorem 1.

Under the condition $m > 2k - 2$ it can be shown that the solution $x (t, t_0, a)$ is exponentially stable under the initial perturbations $| x_0 - a | = O (\varepsilon^{k-1})$. The same is true of the solutions $\xi_m (t, t_0, a)$ and $\xi (t, t_0, a)$ of Eqs. (1. 3) and (1. 9). Thus, a domain of exponential attraction of radius $\delta = O (\varepsilon^r)$, where $r = k - 1$, exists at $t = t_0$. If $k = 1$, then one and the same approximation of order m follows from Theorems 1 and 2 for all $m \geqslant 1$. From Theorem 2 it follows as well that the solution $x (t, t_0, a)$ is exponentially stable. When $k \leqslant m \leqslant 2k - 2$ an approximation on finite intervals of order greater than $1/\varepsilon$ can be obtained from Eq. (1. 22). Using (1. 10) and (1. 11) the results of Theorems 1 and 2 can be extended to the case when the resolving matrix satisfies, instead of the exponential stability condition (1. 6), the condition

$$\| U_m (t, s, t_0, a) \| \leqslant P_\lambda (\varepsilon t) \, e^{-\nu_m (t-s)} \tag{1.25}$$

where $P_\lambda (\varepsilon t)$ is a polynomial of degree λ in εt. In particular, when $\lambda = 1$ (a case typical of damped oscillations in systems with little friction) bound (1. 7) takes the form

$$| x (t) - x^{(m)} (t) | < C_m | \varepsilon |^{m-2k+1}, \quad m > 2k + r - 1$$

2. **Approximation of solutions of linear equations on an infinite time interval.** Consider the linear equation

$$y^{\cdot} = A (x) y + f (x, t) \tag{2.1}$$

where y is a column vector with components $y_1, \ldots, y_p$ and the known n-dimensional vector $x(t, \varepsilon)$ has the derivative $x^{\cdot} = \varepsilon X(t)$ proportional to a small parameter ε. The following procedure is possible for constructing asymptotic approximations to the solution of the Cauchy problem for Eq. (2.1) with initial condition $y(t_0) = b$. We write the approximation $y^{(j)}(t)$ as

$$y^{(j)} = \varphi_0(t, x) + \varepsilon\varphi_1(t, x) + \ldots + \varepsilon^{(j)}\varphi_j(t, x) \tag{2.2}$$

where $\varphi_0(t, x)$ is a solution of the equation

$$\varphi_0^{\cdot} = A(x)\varphi_0 + f(x, t)$$

in which x is taken to be a parameter not depending on t. The subsequent terms in expression (2.2) are determined in succession from the equations resulting from the substitution of (2.2) into (2.1) and from equating the coefficients of like powers of ε. We arrive at the equations

$$\varphi_i^{\cdot} = A(x)\varphi_i - \frac{\partial\varphi_{i-1}}{\partial x} X$$

which we integrate under the assumption that x is a parameter not depending on t. For definiteness we can set $\varphi_0(t_0, x(t_0)) = b$ and $\varphi_i(t_0, x(t_0)) = 0$. The functions φ_i are determined to within an arbitrary function of x, differentiable a sufficient number of times, taking the specified value when $x = x(t_0)$. Such a situation is usual for asymptotic methods.

Theorem 3. Suppose that vector $x(t)$ remains for all $t \geqslant t_0$ in a domain G of space $x_1, \ldots, x_n$. For $x \in G$ and for all $t \geqslant t_0$ let the functions $A(x)$, $f(x, t)$ and $X(t)$ be uniformly bounded, $f(x, t)$ and $X(t)$ be continuous in t, and $A(x)$ and $f(x, t)$ have m uniformly bounded derivatives with respect to x. For $x \in G$ let the eigenvalues $\lambda_\rho(x)$ $(\rho = 1, \ldots, p)$ of matrix $A(x)$ satisfy the condition $\operatorname{Re}\lambda_\rho(x) < -\mu < 0$, where μ is independent of x and ε. Then for sufficiently small $|\varepsilon|$ the solution $y(t)$ is approximated by the approximation $y^{(j)}(t)$ on the whole time interval with accuracy $|\varepsilon|^{j+1}$, i.e., for all $t \geqslant t_0$

$$|y(t) - y^{(j)}(t)| \leqslant K_j|\varepsilon|^{j+1} \tag{2.3}$$

where K_j is independent of t and ε.

To prove this we consider the difference $v_j = y - y^{(j)}$. It satisfies the equation and initial condition

$$v_j^{\cdot} = A(x)v_j - \varepsilon^{j+1}\frac{\partial\varphi_{j-1}}{\partial x} X, \quad v_j(t_0) = 0$$

By Coppel's theorem (see Sect. 5 in Chapter VI of [5], for instance) the resolving matrix $L(t, t_0)$ of the corresponding homogeneous equation satisfies the conditions

$$\| L(t, t_0) \| \leqslant Qe^{-\gamma(t-t_0)}$$

where Q and γ do not depend on t and ε. Therefore,

$$|v_j| = |\varepsilon|^{j+1}\left|\int_{t_0}^{t} L(t, s)\frac{\partial\varphi_{j-1}}{\partial x} X ds\right| \leqslant |\varepsilon|^{j+1} Q\int_{t_0}^{t} e^{-\gamma(t-s)}\left|\frac{\partial\varphi_{j-1}}{\partial x} X\right| ds \tag{2.4}$$

Function φ_0 is a bounded function of t as a consequence of the boundedness of $f(x, t)$ and of the condition $\operatorname{Re}\lambda_\rho < 0$. From the boundedness of the derivatives with respect to x of $f(x, t)$ and $A(x)$ follows the boundedness of function $(\partial\varphi_0 / \partial x) X$. Therefore, function φ_1 is bounded as well, etc. Finally, the function $(\partial\varphi_{j-1} / \partial x) X$, occurring in the integrand in (2.4), is bounded. Hence follows bound (2.3).

In the expressions for the derivatives $\partial\varphi_i / \partial x$ there occur secular terms containing products of functions of the form $t^k \exp \lambda_\rho t$ by bounded time functions. When $X(t)$ and $f(x, t)$ are periodic in t with period independent of x or are finite sums of the form

$$\sum_\nu a_\nu(x) \cos \omega_\nu t + b_\nu(x) \sin \omega_\nu t$$

where the frequencies ω_ν, independent of x, are mutually irrational, while the λ_ρ are real quantities, we can find an algorithm for constructing the asymptotic approximations containing only exponentials and periodic or quasiperiodic functions. To do this we should separately construct a periodic or quasiperiodic solution of the inhomogeneous equation and particular solutions of the homogeneous equation, in the same way as, for example, in the case $x = \tau = \varepsilon t$ (see [6], for instance).

3. **Asymptotic separation of motions on an infinite time interval in quasilinear systems with many fast variables.** Consider the quasilinear system

$$x^\cdot = \varepsilon X(x, y, t, \varepsilon), \quad y^\cdot = A(x)y + f(x, t) \tag{3.1}$$

System (3.1) is a special case of the systems with many fast variables studied in [7]. However, for the asymptotic integration of systems of type (3.1) it is more convenient to apply, instead of Volosov's method, a simpler method proposed in [8] especially for quasilinear systems. Using the results obtained in [8] we show that the exact solution of system (3.1) can be approximated by an approximate solution on an infinite time interval. According to [8] an approximate solution of system (3.1) is constructed as follows. Let the initial conditions $x(t_0) = a$ and $y(t_0) = b$ be given. We write $y^{(j)}$ as

$$y^{(j)} = \varphi_0(t, x) + \varepsilon\varphi_1(t, x) + \ldots + \varepsilon^i\varphi_j(t, x) \tag{3.2}$$

Here φ_0 is a solution of the equation

$$\varphi_0^\cdot = A(x)\varphi_0 + f(x, t) \tag{3.3}$$

found under the assumption that in this equation x is a parameter not depending on time. For definiteness we assume that $\varphi_0(t_0, x(t_0)) = \varphi_0(t_0, a) = b$. Then $\varphi_i(t_0, a) = 0,\ i = 1, \ldots, j$.

Entering (3.2) into the first equation in (3.1) and expanding the function $X(x, y^{(j)}, t, \varepsilon)$ in powers of ε, we have

$$x^\cdot = \varepsilon X_0(x, \varphi_0, t) + \varepsilon^2 \left[X_1(x, \varphi_0, t) + \frac{\partial X_0(x, \varphi_0, t)}{\partial\varphi_0} \varphi_1 \right] + \cdots \tag{3.4}$$

Substituting $y^{(j)}$ in the place of y in the second equation in (3.1) and replacing $x^\cdot$ by expression (3.4), by comparing the coefficients of like powers of ε we obtain equations for the successive determination of the functions $\varphi_1(t, x), \ldots, \varphi_j(t, x)$

$$\varphi_1^\cdot = A(x)\varphi_1 - \frac{\partial\varphi_0}{\partial x} X_0(x, \varphi_0, t) \tag{3.5}$$

$$\varphi_2^\cdot = A(x)\varphi_2 - \frac{\partial\varphi_1}{\partial x} X_0(x, \varphi_0, t) - \frac{\partial\varphi_0}{\partial x} \left[X_1(x, \varphi_0, t) + \frac{\partial X_0}{\partial\varphi_0} \varphi_1 \right]$$

etc. Equations (3. 5) are integrated under the condition that $x = \text{const.}$ Inserting the functions $y^{(j)}(t, x)$ thus found in the place of y in the first equation in (3. 1), we arrive at a system in standard form

$$x^{\cdot} = \varepsilon X(x, y^{(j)}(t, x, \varepsilon), t, \varepsilon) \tag{3. 6}$$

to which we can apply the averaging method. As we shall see from what follows, when determining the m-th approximation to the solution of system (3. 1) it makes sense to examine system (3. 6) only for $j = m - 1$

Theorem 4. For $x \in G$, $y \in G_1$, $|\varepsilon| \leqslant \varepsilon_0$ and $t \geqslant t_0$ let the functions $f(x, t)$, $A(x)$ and $X(x, y, t, \varepsilon)$ satisfy with respect to variables x, t, ε the same requirements as in Sects. 1 and 2. Let function X have $m + 1$ uniformly bounded derivatives with respect to y. For $j = m - 1$ let the improved m-th approximation to the solution of system (3. 6) and the equation for the slow motions, obtained from (3. 6), possess the same properties as in Theorem 1 and let $y^{(m-1)}(t, x^{(m)}, \varepsilon)$ remain in $G_1 - \alpha$, where α is independent of t and ε. Then the solution of system (3. 1) with initial conditions

$$x(t_0) = x^{(m)}(t_0) = \xi_m(t_0) = a, \quad y(t_0) = y^{(m-1)}(t_0, a) = b$$

remains in $G \times G_1$ for all $t \geqslant t_0$ for sufficiently small $|\varepsilon|$ and $m > k + r - 1$ and can be approximated by the functions $x^{(m)}$ and $y^{(m-1)}(t, x^{(m)})$ with an accuracy $|\varepsilon|^{m-k+1}$, i.e.,

$$|x - x^{(m)}| < C_m |\varepsilon|^{m-k+1}, \tag{3. 7}$$

$$|y - y^{(m-1)}(t, x^{(m)})| < C_{1m} |\varepsilon|^{m-k+1}$$

Proof. We assume that x and y and all their approximations being examined remain in G and G_1. In (3. 1) we introduce a new variable ψ by the relation

$$\psi = y - y^{(m-1)}(t, x) \quad (\psi(t_0) = 0) \tag{3. 8}$$

We arrive at the system

$$x^{\cdot} = \varepsilon X(x, y^{(m-1)}, t, \varepsilon) + \varepsilon P_m(x, \psi, t, \varepsilon)\psi \tag{3. 9}$$

$$\psi^{\cdot} = A(x)\psi - \varepsilon\left(\frac{\partial\varphi_0}{\partial x} + \ldots + \varepsilon^{m-1}\frac{\partial\varphi_{m-1}}{\partial x}\right)P_m\psi + \varepsilon^m\Psi_m$$

Here $P_m\psi$ is the remainder term in the Lagrange formula representation of $X(x, y^{(m-1)} + \psi, t, \varepsilon)$ while Ψ_m consists of terms of order ε^m and higher in the expression $\varepsilon(\partial y^{(m-1)} / \partial x)X(x, y^{(m-1)}, t, \varepsilon)$ if X is represented as an expansion in powers of ε with a remainder term of order $m - 1$. By virtue of the theorem's hypothesis the functions P_m, Ψ_m, $\partial\varphi_0 / \partial x, \ldots, \partial\varphi_{m-1} / \partial x$ are uniformly bounded. Thereore, for sufficiently small $|\varepsilon|$ the real parts of the eigenvalues of the matrix

$$A(x) - \varepsilon(\partial\varphi_0 / \partial x + \ldots + \varepsilon^{m-1}\partial\varphi_{m-1} / \partial x)P_m$$

are less than the constant $-\mu' = -\mu + |\varepsilon\mu_1|$. This permits us to apply Coppel's theorem to the second equation in (3. 9) and to obtain, analogously to Sect. 2, the bound $|\psi| \leqslant c|\varepsilon|^m$, where c is independent of t and ε. Thus, we obtain the bound $|\varepsilon P_m\psi| \leqslant p_{1m}|\varepsilon|^{m+1}$ for the term $\varepsilon P_m\psi$.

Suppose that for $j = m - 1$ the improved m-th approximation of form (1. 2) and the Eq. (1. 3) have been constructed for system (3. 6). In (3. 9) we introduce the new variable ξ by the relation (1.8). Instead of the first equation in (3.9) we obtain the equation, analogous to (1.9),

$$\frac{d\xi}{dt} = \varepsilon\Xi^{(m-1)}(\xi) + \varepsilon^{m+1}[R_m(\xi, t, \varepsilon) + R_{1m}(\xi, \psi, t, \varepsilon)] \qquad (3.10)$$

where the function R_{1m} is uniformly bounded. Now we can repeat the proof of Theorem 1, replacing R_m by $R_m + R_{1m}$. Hence follows the first of bounds (3.7).

Consider the expression $y^{(m-1)}(t, x^{(m)}(t), \varepsilon)$. We obtain

$$| y - y^{(m-1)}(t, x^{(m)}, \varepsilon) | = | [y - y^{(m-1)}(t, x, \varepsilon)] +$$
$$[y^{(m-1)}(t, x, \varepsilon) - y^{(m-1)}(t, x^{(m)}, \varepsilon)] | \leqslant$$
$$c | \varepsilon |^m + h | x - x^{(m)} | \leqslant c | \varepsilon |^m + hC_m | \varepsilon |^{m-k+1}$$

Here the constant h, independent of t, and ε, exists by virtue of the boundedness of the derivative $\partial y^{(m-1)} / \partial x$. The second bound in (3.7) is obtained from (3.10). Having the bounds derived above we can show, analogously to Sect. 1, that from the conditions $\xi_m(t) \in G$ and $y^{(m-1)}(t, x^{(m)}) \in G_1 - \alpha$ it follows that for sufficiently small $| \varepsilon |$ the functions $\xi(t)$, $x(t) \in G$ and $y(t) \in G_1$.

If the hypotheses of Theorem 2 are valid for the Eqs. (3.6) of slow motions when $j = m - 1$, then an approximation of form (3.7) can be proved when $m > 2k - 2$. It turns out that the solution $x(t)$ is exponentially stable, while the domain of attraction at $t = t_0$ has dimensions of $O(| \varepsilon |^{k-1})$ with respect to x and dimensions not depending on ε with respect to y.

Another variant is possible for eliminating the fast variables in system (3.1). Now x is written in form (1.2) and y as

$$y^{(m-1)} = \varphi_0(t, \xi_m) + \varepsilon\varphi_1(t, \xi_m) + \ldots + \varepsilon^{m-1}\varphi_{m-1}(t, \xi_m)$$

An equation of form (1.3) is constructed for ξ_m. Inserting the expression indicated into Eqs. (3.1), replacing $\xi^{\cdot}$ in accordance with (1.3), and equating terms of like powers of ε, we obtain the equation

$$\varphi_0^{\cdot} = A(\xi_m)\varphi_0 + f(\xi_m, t), \quad \xi_m = \text{const}$$

for φ_0. After this the function Ξ_0 is found as the average of X with $y = \varphi_0$, etc. In other words, in this variant the averaging of system (3.6) and the representation of the fast variables in terms of slow ones are implemented simultaneously. An approximation of type (3.7) can be proved for the second variant as well, with the sole proviso that the functions $Z_m = \xi - \xi_m$ and $\psi = y - y^{(m-1)}$ be estimated simultaneously.

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